VANISHING BERGMAN KERNELS

Antti Perälä
University of Barcelona

Bergman kernel

Let Ω be a domain in $\mathbb{C}^n$ and ω a non-negative measurable function (a weight). Denote by $A^2(\Omega, \omega)$ the space of holomorphic functions $f : \Omega \to \mathbb{C}$ with

$$\|f\|_{\Omega,\omega} = \left(\int_{\Omega} |f(\xi)|^2 \omega(\xi) dV_n(\xi) \right)^{1/2} < \infty.$$

- ω is called admissible (or Bergman weight), if for every compact $K \subset \Omega$, there exists C_K so that $|f(z)| \leq C_K \|f\|_{\Omega,\omega}$ uniformly for $z \in K$
- This implies that $A^2(\Omega, \omega)$ is a Hilbert space with respect to

$$\langle f, g \rangle_{\Omega,\omega} = \int_{\Omega} f(\xi) \overline{g(\xi)} \omega(\xi) dV_n(\xi)$$

- There exists a unique $B_z^{\Omega,\omega} \in A^2(\Omega, \omega)$ with

$$f(z) = \langle f, B_z^{\Omega,\omega} \rangle_{\Omega,\omega}$$

- $B_z^{\Omega,\omega}$ is called the Bergman kernel

Examples

- $B_z^{\mathbb{D},1}(\xi) = (1-\bar{z}\xi)^{-2}$; $\mathbb{D} = \{|z|<1\}$ unit disk of $\mathbb{C}$
- $B_z^{\Omega,1}(\xi) = \overline{\varphi'(z)}\varphi'(\xi)(1-\overline{\varphi(z)}\varphi(\xi))^{-2}$; Ω simply connected and $\varphi : \Omega \to \mathbb{D}$ the Riemann map
- $B_z^{\mathbb{B}_n,\nu_\alpha}(\xi) = (1-\langle \xi, z\rangle)^{-2-\alpha-n}$; $\mathbb{B}_n$ the unit ball of $\mathbb{C}^n$ and $\nu_\alpha(z) = c_{n,\alpha}(1-|z|^2)^\alpha$ $(\alpha > -1)$ a standard weight
- $B_z^{\mathbb{C}^n,\lambda_\gamma}(\xi) = e^{\gamma\langle \xi, z\rangle}$; $\lambda_\gamma(z) = e^{-\gamma|z|^2}$ the Gaussian (Fock) weight
- In general, if $(e_k)_{k\geq 0}$ is an orthonormal basis of $A^2(\Omega,\omega)$, we have $B_z^{\Omega,\omega}(\xi) = \sum_{k\geq 0} \overline{e_k(z)} e_k(\xi)$

Notation

$$K^{\Omega,\omega}(z,\xi) = B_z^{\Omega,\omega}(\xi)$$

None of the kernels above have zeroes.

Non-vanishing kernels

Since $f(z) = \langle f, B_z^{\Omega,\omega} \rangle_{\Omega,\omega}$, it clear that

$$|f(z)| \leq \|f\|_{\Omega,\omega} \|B_z^{\Omega,\omega}\|_{\Omega,\omega} = \|f\|_{\Omega,\omega} B_z^{\Omega,\omega}(z)^{1/2},$$

and this bound is sharp. Moreover $B_z^{\Omega,\omega}$ itself is extremal for the functional $f \mapsto f(z)$.

Is there an L^p version of this result?

If $f \mapsto f(z)$ is L^p bounded, then Hahn-Banach guarantees existence of $F_z^{\Omega,\omega}$, whose $L^{p'}$ norm agrees with the norm of the functional.

If $B_z^{\Omega,\omega}$ has no zeroes

$$F_z^{\Omega,\omega}(\xi) = B_z^{\Omega,\omega}(\xi) \overline{B_z^{\Omega,\omega}(\xi)}^{2/p'-1} B_z^{\Omega,\omega}(z)^{1-2/p'},$$

and the extremal is simply $(B_z^{\Omega,\omega})^{2/p}$.

For vanishing kernels, this argument falls apart.

Another look at the point evaluation

Let A^p_α be the standard weighted Bergman space with respect to the weight $(\alpha+1)(1-|z|^2)^\alpha$. Denote by

$$\varphi_z(\xi) = \frac{z-\xi}{1-\overline{z}\xi}$$

the Möbius automorphism of the disk. Clearly the map

$$f \mapsto (f\circ\varphi_z)(\varphi_z')^{(2+\alpha)/p}$$

is an isomorphic isomorphism $A^p_\alpha \to A^p_\alpha$. This fact can be used to deduce the sharp inequality

$$|f(z)| \le \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}.$$

This result is due to D. Vukotić (1993).

Trivial case

Classification of domains in $\mathbb{C}^n$ with vanishing Bergman kernels is known as the Lu Qi-Keng's problem. (Lu Qi-Keng 1966)

If ω has a non-integrable singularity at some point $\xi_0 \in \Omega$, then all functions $f \in A^2(\Omega, \omega)$ must vanish at that point. In particular, for every $z \in \Omega$, $B_z^{\Omega,\omega}(\xi_0) = 0$.

- Typical example: Ψ holomorphic with zeroes, and $\omega = |\Psi|^{-2}$ (might have any finite number, or even infinitely many zeroes)
- Easy to see, because Bergman kernels are unique
- Often we want $\omega \in L^1$

Can this reasoning lead to an L^1 weight?

Variations of a theorem of Ramadanov

Theorem. (Ramadanov 1967)

Suppose Ω and Ω_j are domains in $\mathbb{C}^n$ so that $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega$ and $\bigcup_j \Omega_j = \Omega$. Then $K^{\Omega_j.1} \to K^{\Omega,1}$ uniformly on compact subsets of $\Omega \times \Omega$.

Forelli-Rudin construction (Forelli, Rudin 1976, Ligocka 1989)

Let D be a domain in $\mathbb{C}^n$, ω a continuous weight on D, and Ω the Hartogs domain

$$\Omega = \{(z, z') \in D \times \mathbb{C}^m : \|z'\|^{2m} < \omega(z)\}.$$

Then $B_z^{D,\omega}(\xi) = const\, B_{(z,0)}^{\Omega,1}(\xi, 0)$.

Theorem (simplified). (Pasternak-Winiarski, Wójcicki 2016)

Suppose ω and ω_j admissible weights on $\Omega \subset \mathbb{C}^n$ so that $\omega_1 \leq \omega_2 \leq \dots \omega$. If $\omega_j \to \omega$ almost everywhere, then $K^{\Omega,\omega_j} \to K^{\Omega,\omega}$ uniformly on compact subsets of $\Omega \times \Omega$.

Vanishing Bergman kernels

Building on the aforementioned theorem of Ramadanov, we have

Theorem. Boas (1996)

"Most" domains in $\mathbb{C}^n$ have vanishing (unweighted) Bergman kernels.

There is also a very strong weighted variant of this result:

Theorem. Boudreaux (2017)

Let Ω be a domain in $\mathbb{C}^n$ and ω an admissible weight. There exists an admissible weight ω' so that $K^{\Omega,\omega'}$ has zeroes and $A^2(\Omega,\omega) = A^2(\Omega,\omega')$. Moreover, any finite number of zeroes can be prescribed and if $\omega \in L^1$, we may have $\omega' \in L^1$.

Idea of proof:

Modify ω by adding any finite number of non-integrable singularities. Approximate this weight by L^1 weights and use the weighted Ramadanov theorem of Pasternak-Winiarski and Wójcicki. Theorem of Hurwitz guarantees that some of the approximating weights have kernels with the claimed property.

Remarks

There are some problems with this approach:

- Does not give explicit weight
- Does not give explicit kernel
- Does not allow infinite number of zeroes

Theorem. Zeytuncu (2011)

Let $W(z) = 18$ when $|z| < 1/4$ and $W(z) = 1$ when $|z| \geq 1/4$. The kernel $K^{\mathbb{D},W}$ has zeroes.

- Note that W is radial: $W(z) = W(|z|)$
- For radial ω monomials form orthonormal basis for $A^2(\mathbb{D}, \omega)$
- For radial ω, $B_z^{\mathbb{D},\omega}(\xi) = \sum_{k\geq 0} \frac{1}{\omega_k}(\xi\overline{z})^k$, where the numbers ω_k are moments of ω
- No closed form formula for $K^{\mathbb{D},W}$

Various classes of radial weights on the disk have been studied Peláez and Rättyä in a series of papers.

Littlewood-Paley formula

Let ω be a radial weight in L^1

$$\omega^*(r) = \int_r^1 \omega(s) s \log(s/r) ds$$

$$\langle f, g \rangle_{\mathbb{D},\omega} = 4\langle f', g' \rangle_{\mathbb{D},\omega^*} + \omega(\mathbb{D}) f(0)\overline{g(0)}$$

Let now $F' = f$ with $F(0) = 0$. Then

$$f(z) = F'(z) = \langle F, \partial_{\overline{z}} B_z^{\mathbb{D},\omega} \rangle_{\mathbb{D},\omega}$$

An application of Littlewood-Paley formula together with the uniqueness of the Bergman kernel now give

$$K^{\mathbb{D},\omega^*}(z,\xi) = \partial_\xi \partial_{\overline{z}} K^{\mathbb{D},\omega}(z,\xi)$$

Vanishing Bergman kernels on the disk

Recall that for $\alpha > -1$ the weight $\nu_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$ gives rise to the non-vanishing kernel $B_z^{\mathbb{D},\nu_\alpha}(\xi) = (1-\bar{z}\xi)^{-2-\alpha}$.

$$B_z^{\mathbb{D},\nu_\alpha^*}(\xi) = 4(2+\alpha)\frac{1+(2+\alpha)\bar{z}\xi}{(1-\bar{z}\xi)^{4+\alpha}}$$

has exactly one zero if z is big enough.

Can we iterate $\omega \mapsto \omega^*$ to get more zeroes?

$$B_z^{\mathbb{D},\nu_0^{*2}}(\xi) = const\,\frac{1+6\bar{z}\xi+3(\bar{z}\xi)^2}{(1-\bar{z}\xi)^6}$$

has only one zero in $\mathbb{D}$.

Nevertheless, we have the following:

Vanishing Bergman kernels on the disk

Theorem. Perälä (2017)

Let $n \in \mathbb{N}$. There exists a weight $\omega_{(n)}$ and a standard weight ν_β so that $A^2(\mathbb{D}, \omega_{(n)}) = A^2(\mathbb{D}, \nu_\beta)$ and $B_z^{\mathbb{D},\omega_{(n)}}$ has exactly n zeroes, provided that z is large enough.

Idea of proof:

Note first that

$$B_z^{\mathbb{D},\nu_\alpha^{*n}}(\xi) = \frac{p_{\alpha,n}(\overline{z}\xi)}{(1-\overline{z}\xi)^{2+\alpha+2n}},$$

where $p_{\alpha,n}$ is a polynomial of degree n. A more careful analysis of $p_{\alpha,n}$ shows that, if α is large enough, the highest (that is nth) order term in $p_{\alpha,n}$ dominates the rest of $p_{\alpha,n}$ on the boundary of the unit disk. The proof is completed by an application of Rouché's theorem.

It is possible to calculate some suitable α, given n.

Remarks

- If ω is radial, the formula $B_z^{\mathbb{D},\omega}(\xi) = \sum_{k\geq 0} \frac{1}{\omega_k}(\xi\overline{z})^k$ shows that its kernel (on the disk) can never have infinitely many zeroes.
- The same argument applied to the Gaussian weight $e^{-\gamma|z|^2}$ (on the complex plane) produces kernels with finite prescribed number of zeroes.
- Let $M_\alpha(z) = e^{-|z|^\alpha}$. The kernel $B_z^{\mathbb{C},M_\alpha}$ can be expressed in terms of the Mittag-Leffler functions. It follows that (for instance) the kernel has infinitely many zeroes if $1 < \alpha < 2$.

Question

Does there exist an L^1 weight ω on the disk, so that its kernel has infinitely many zeroes?

- Impossible if ω is radial
- If Ψ is a non-trivial analytic function with infinitely many zeroes, the kernels of $\omega = |\Psi|^{-2}$ has infinitely many zeroes, but ω is not L^1

THANK YOU!